Chapter 4
Linear Programming Problem

Graphical method of its solution. Application to real life problems.

4.1 Introduction:

Programming problems are generally concerned with the allocation of scarce resources. These resources include labour, capital, machines, material, etc. The resources are to be allocated in such a way as to minimize costs or maximize profits. Linear Programming Problem (LPP) is one of the types of programming problem. It is a tool for solving optimization (minimization or maximization) problems. LPP is applied in various areas such as industries, banking, education, forestry etc.

4.2 Definition of terms and statement of L.P.P

(i) Decision variables: A set of variables which are present in the objective function and the constraints of an L.P.P are called decision variables. The decision variables are denoted by $x_1, x_2, ..., x_n$.

(**ii) Objective function:** A linear function of the decision variable which is required to be optimized (either maximized or minimized) is called an objective function of an L.P.P and is denoted by $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$, where the constants $c_1, c_2, ..., c_n$ are known as technological coefficients.

(iii) Constraints: A set of linear equations/inequalities of the decision variables is called constraints. While solving a L.P.P these constraints are not to be violated.

(iv) Non-negative restrictions: In a L.P.P, the decision variables $x_1, x_2, ..., x_n$ have to be non negative. In other words, we must have $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$. This requirement is called non-negative restrictions.

(v) Statement of a L.P.P.: A linear programming problem is a mathematical problem that is concerned with optimizing (maximizing or minimizing) a linear function of the decision variables subject to a system of linear constraints (equalities or inequalities) and non-negativity restrictions of the decision variables.

A general linear programming problem may be algebraically stated as below:

$$\text{Optimize } z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

Subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\leq = \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\leq = \geq b_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\leq = \geq b_m
\end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, \ldots x_n \geq 0.$$

Note in each of the above m constrains one and only of the equality/inequality $\leq$ or $=$ or $\geq$ holds. The known constants $b_1, b_2, ..., b_m$ are the available resources or requirements. In general, we may represent a L.P.P in matrix notation.

(vi) Solution of a L.P.P: Solving a L.P.P. means finding a set of decision variables which optimize the objective function and satisfy the given set of constrains and non-negativity restrictions.

4.3 Formulation of a L.P.P

The steps involved in formulation of a L.P.P are the following:

(i) Identify the decision variables $x_1, x_2, ..., x_n$.

(ii) Formulate the objective function $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ which is to be optimized

(iii) Identify the set of constrains which are not to be violated and the non-negativity restrictions.

4.4 Steps involved in solving a L.P.P in two variables by graphical method

We go through the following steps while solving a L.P.P. in two variables by the graphical method.

Step 1: Formulate the L.P.P using the given information

Step 2: Represent the constraints of the L.P.P. in the (x-y) plane using a graph.

Step 3: Identify the feasible region and the vertices of the convex region corresponding to the feasible region

Step 4: Evaluate the value of the objective function at these vertices

Step 5: Find the optimum (maximum or minimum) value of the objective function and the value of the corresponding decision variable at this vertex.

We illustrate the ideas of linear programming with the help of an example below.

Example 1

A furniture company makes and sells tables and chairs. Each table requires 5 board-feet of oak, 2 board-feet of pine, and 4 labour-hours. Each chair requires 2 board-feet of oak, 3 board-feet of pine and 3 labour hours. There are 122 board feet of oak, 100 board feet of pine and 120 labour hours available for the upcoming week. How many tables and chairs should the company make during the next week so as to maximize its profit given that each table makes a profit of $P12$ and each chair makes a profit of $P8$?

Solution:

There are many combinations of table and chair production. The problem is which combination to choose in order to make maximum profit.

Table 1: Resources required per product unit

	Oak board- feet	Pine board -feet	Labour hours	Profit per unit
Tables	5	2	4	12

Chairs	2	3	3	8
Available resources	122	100	120	

This problem can be solved by formulating equations and inequalities using the data available.

Let x be the number of tables to produce next week and let y be the number of chairs to produce next week.

Using these variables, the profit function to be formed is as follows:

$$Z = 12x + 8y \qquad (1)$$

x and y in (1) are called **decision variables.** In order to solve our problem we have to take decisions on these variables to determine the values they take. Equation (1) is called the **objective function** because it describes our objective in solving the problem. In order to maximize the profit we have to maximize $Z = 12x + 8y$, subject to the restrictions on x and y. These restrictions are called **constraints**.

We need to use 5 board-feet of oak for each table and 2 board-feet of oak for each chair. The available oak board-feet is 122. Thus, if we are producing x tables and y chairs then the total oak board-feet needed is $5x + 2y$ and this should not exceed 122, or mathematically, $5x + 2y \le 122$.

Similarly, the available restrictions on pine board-feet and labour hours results in $2x + 3y \le 100$ and $4x + 3y \le 120$ respectively.

Thus, the three constraints in producing x tables and y chairs are

$5x + 2y \le 122$, $2x + 3y \le 100$, $4x + 3y \le 120$.

The coefficients of the decision variables in the constraints are called **technological coefficients**.

In addition to these, we have to put non negativity constraints on the variables themselves, as they are number of tables and chairs. That is $x \ge 0, \quad y \ge 0$.

These are called **non-negativity constraints**.

In summary the linear programming model is:

$$\text{Maximize } Z = 12x + 8y$$

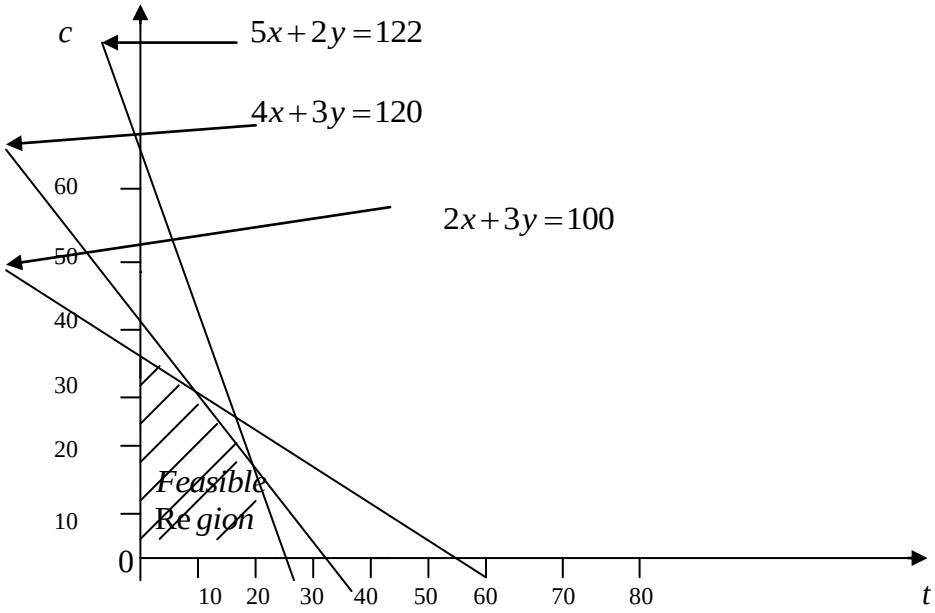
Subject to

$$5x + 2y \le 122$$
$$2x + 3y \le 100$$
$$4x + 3y \le 120$$

$$x \ge 0 \text{ and } y \ge 0.$$

Suppose we try to represent the constraints of Example 7.1 graphically. Each less than or equal to resource constraint and each non negativity restriction establishes a boundary constraint for the objective function. When all boundaries are considered together, the resulting area consisting of points corresponding to possible solution values for the problem. This region is referred to as the region of feasible solutions.

It may be noted that the non negativity constraint restricts to first quadrant.



The shaded region satisfies the constraints of the given problem. Therefore, the solution to the problem lies in the shaded region.

It may be proved mathematically (using the theory of convex sets) that the solution lies on one of the vertices of the shaded region (proof is avoided at this stage).

Let us look into the vertices of this shaded region. There are 5 vertices altogether. The coordinates and the corresponding value of the objective function are given below.

Vertex	Coordinates	P
A	$(0, 33.33)$	266.64
B	$(10, 26.67)$	333.36
C	$(18, 16)$	344^{*}
D	$(24.4, 0)$	292.80
E	$(0, 0)$	0

It can be seen that the optimum solution is reached at $x = 18$ and $y = 16$. In other words the maximum profit is achieved at the said quantities above.

Example 2: Diet Mix problem**:**

Solution

A dog food seller decided that his diet mix must contain at least 10 units of proteins, 5 units of carbohydrates, and 12 units of vitamins. There are two ingredients that he can use to obtain the desired mix. Each pound of ingredient A contains 2 units of proteins, 2 units of carbohydrates and 2 units of vitamins. Each pound of ingredient B contains 3 units of proteins, 1 unit of carbohydrates and 4 units of vitamins. Ingredient A costs

$P0.25$ per pound and B costs $P0.30$ per pound. How would the mix be made so that it has sufficient proteins, carbohydrates and vitamins and its cost is as low as possible?

Solution:

Table 7.2

	units/pound A	units/pound B	Minimum requirements
Proteins	2	3	10
Carbohydrates	2	1	5
Vitamins	2	4	12
Cost per pound	$P0.25$	$P0.30$	

Here the problem is to determine the amount of each ingredient (A and B) so that the resulting mix has sufficient proteins, carbohydrates and vitamins and its cost is as low as possible.

Let a be the number of pounds of ingredient A to use and b be the number of pounds of ingredient B to use.

The objective function is $C = 0.25a + 0.30b$ and

the constraints are: $2a + 3b \geq 10$, $2a + b \geq 5$, $2a + 4b \geq 12$, $a \geq 0$, $b \geq 0$.

So the linear programming model is:

Minimize $C = 0.25a + 0.30b$

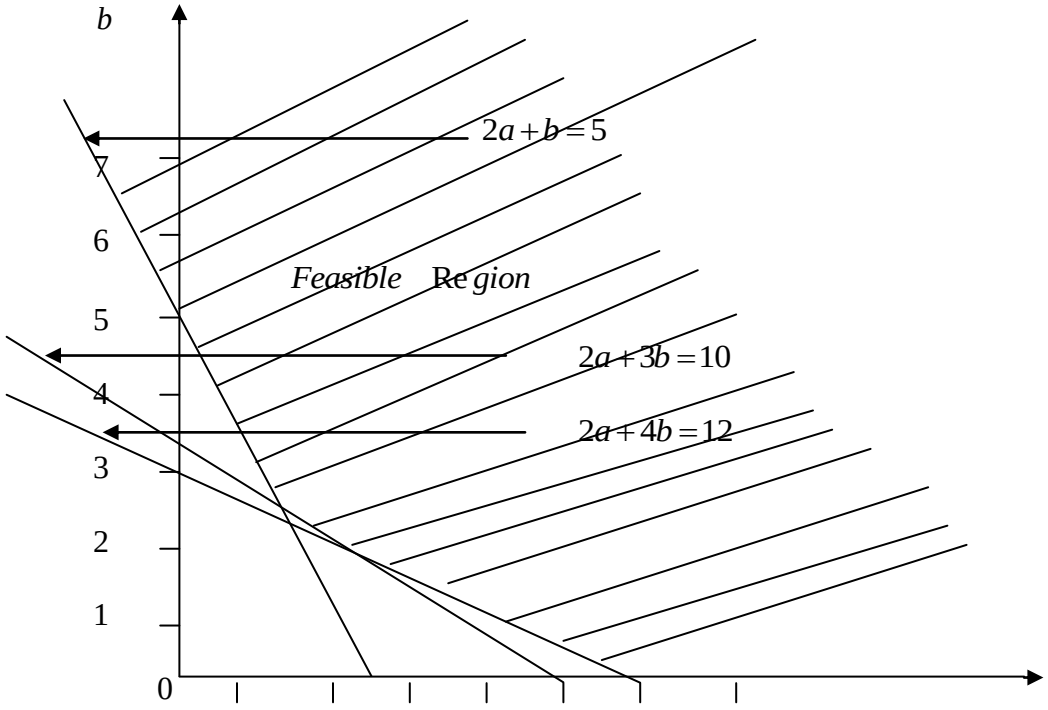
Subject to $2a + 3b \geq 10$

$2a + b \geq 5$

$2a + 4b \geq 12$

$a \geq 0, \quad b \geq 0.$

The graphical solution may be obtained as below.



1 2 3 4 5 6 7 *a*

Vertex	Coordinates	C
A	$(0, 5)$	1.5
B	$\left(\frac{5}{4}, \frac{5}{2}\right)$	1.06*
C	$(2, 2)$	1.1
D	$(6, 0)$	1.5

The optimum solution is attained at $a = \frac{5}{4}$ and $b = \frac{5}{2}$. That is, the cost function is minimum at $a = \frac{5}{4}$ and $b = \frac{5}{2}$.

Note: In our discussion, we have considered LP problems with two decision variables. In a practical situation, there can be several variables. However, graphical representation is possible only in the case of a 2-dimensional problem. One has to employ other methods to solve the LP problems of higher dimensions. One such important method is **Simplex Algorithm.**

DO IT YOURSELF EXERCISES

1. Solve the following linear programming problems:

(i) Maximize $Z = 3x + 4y$ subject to $x + 2y \le 12,\ 5x + y \le 15,\ x \ge 0, y \ge 0$.
(ii) Maximize $Z = 3x + 7y$ subject to
$x + 2y \le 16,\ 4x + y \le 22,\ 2x + 8y \le 34,\ x \ge 0,\ y \ge 0$.
(iii) Minimize $Z = 4x + 6y$ subject to $x + y \ge 20,\ x \ge 10,\ y \ge 6$.
(iv) Minimize $C = 2x + 5y$ subject to $3x + y \ge 18,\ x + 2y \ge 16,\ x \ge 0,\ y \ge 0$.

2. An ice cream manufacturer blends ingredients A, B and C in varying proportions to make two types of ice cream: Regular (R) and Extra rich (E). Each litre of R contains 0.1 litre of A, 0.2 litre of B, and 0.4 litre of C. Each litre of E contains 0.5 litre of A, 0.2 litre of B, and 0.1 litre of C. The profits on ice cream R and E are P40 and P20 per litre respectively. If the manufacturer has 50 litres of A, 80 litres of B, and 100 litres of C available, how many litres of each type of ice cream should be made to maximize the profit?

3. Joe Armstrong has 250 acres of land on which to plant corn and soybeans. Each acre of corn requires 4 units of storage space and 2 hours of labour. Each acre of soybeans requires 5 units of storage space and 9 hours of labour. Corn returns P140 profit per acre and soybeans return P150 per acre. Given that there are 1500 labour hours and 600 units of storage space, how many acres should be devoted to each crop?

4. A furniture manufacturer makes two types of chairs – one that reclines and other one that does not. The recliner chair requires 2 hours assembly time and 1 hour of packing time. The non-recliner chair requires 1 hour of assembly and 1 hour of packing time. Each month, the manufacturer has 420 work-hours available for assembly and 300 work-hours for packing. The profit on each recliner chair is P140 and that on each non-recliner chair is P100. How many chairs of each type should be produced per month in order to maximize profit?

5. A rancher has 1500 acres of grazing pasture to raise sheep and cattle. The rancher has the resources to raise no more than 4800 animals. Each acre of pasture sustains 2 cows or 6 sheep. The profit on each acre of gazing cow is P80 and that on sheep is P90. How many acres should be allotted for cattle and sheep in order to maximize the profit?

6. A pet-store owner has decided to make and sell dog food. He has decided that the mix must contain at least 10 units of protein, 5 units of carbohydrates and 12 units of vitamins. There are two ingredients that he can use to obtain the desired mix. Each kilogram of ingredient A contains 2 units of protein, 2 units of carbohydrates and 2 units of vitamins. Each kilogram of ingredient B contains 3 units of protein, 1 unit of carbohydrates and 4 units of vitamins. Ingredient A costs P0.25 per kilogram and ingredient B costs P0.30 per kilogram. How should the mix be made so that it has sufficient protein, carbohydrates and vitamins, and its cost is as low as possible?